Journal of Sound and Vibration (1997) **201**(3), 404–406

RELATIONSHIP BETWEEN FUNDAMENTAL NATURAL FREQUENCY AND MAXIMUM STATIC DEFLECTION FOR ROTATING TIMOSHENKO BEAMS

V. T. Nagaraj

Helicopter Design Bureau, Hindustan Aeronautics Limited, Bangalore 560 017, *India*

(*Received* 21 *March* 1996, *and in final form* 8 *July* 1996)

1. INTRODUCTION

Bert [1] proposed a simple relationship between fundamental natural frequency and maximum static deflection for linear systems. Bert's formula is

$$\omega = C(g/\delta)^{1/2}, \tag{1}$$

where C is a dimensionless constant, g is the acceleration due to gravity, and δ is the maximum deflection due to gravity. In [2] Bert showed that equation (1) is valid for Timoshenko beams as well and showed that

$$C = C_0(\omega/\omega_{BE})(\delta/\delta_{BE})^{1/2}, \tag{2}$$

where the subscript BE refers to the Bernoulli–Euler theory and C_0 is the corresponding value of C. In the present Note, it is shown that equation (2) also holds for rotating Timoshenko beams.

2. ANALYSIS

The strain energy and the kinetic energy of a rotating Timoshenko beam are [3]:

$$U = (EI/2L)\int_0^1 [(w' - \psi)^2/s + \psi'^2 + \alpha^2(1 - \eta^2)/2]\,\mathrm{d}\eta, \tag{3}$$

$$T = (mL^3/2)\int_0^1 (\dot{w}^2 + r\dot{\psi}^2)\,\mathrm{d}\eta, \tag{4}$$

where EI and m are the bending rigidity and mass per unit span, respectively. w is the total deflection, ψ is the angle of rotation due to bending, η is the non-dimensional span and $\alpha^2 = (m\Omega^2L^4/EI)$ with Ω the angular speed of rotation. To determine the fundamental frequency of vibration, Rayleigh's quotient will be used with the trial functions derived as follows. The differential equations of motion are:

$$(w' - \psi)'/s + \alpha^2/2((1 - \eta^2)w')' + \lambda^2 w = 0, \tag{5}$$

$$s\psi'' + (w' - \psi) + sr^2\lambda^2\psi = 0. \tag{6}$$

In equation (5) and (6), λ^2 is the non-dimensional frequency and

$$s = EI/KGAL^2 \quad r^2 = I/mL^2. \tag{6a}$$

Eliminating λ^2 from (5) and (6) one gets

$$(s\psi'' + w' - \psi)w + [(w' - \psi)'/s + \alpha^2/2[(1 - \eta^2)w']']sr^2\psi = 0. \tag{7}$$

0022–460X/97/130404 + 03 $25.00/0/sv960733

Since the influence of the rotary inertia term on the frequency is small, the second term in equation (7) can be neglected and one gets:

$$w' = \psi - s\psi'' \tag{8}$$

(The influence of r will be accounted for while calculating the frequency.) Assuming

$$\psi = a_0 + a_1\eta + a_2\eta^2 + a_3\eta^3 \tag{9}$$

and using equation (8) and imposing the cantilever boundary conditions gives

$$w = A[3\eta^2/2 - \eta^3 + \eta^4/4 + s(6\eta - 3\eta^2)], \qquad \psi = A(3\eta - 3\eta^2 + \eta^3). \tag{10, 11}$$

Using Rayleigh's method, equations (3), (4), (9), and (10) give

$$\lambda^2 = m\omega^2 L^4/EI = K/M_D, \tag{12}$$

where

$$K = 1{\cdot}8 + 12s + \alpha^2(0{\cdot}1694 + 1{\cdot}3571s + 5{\cdot}4s^2), \tag{13}$$

$$M_D = 0{\cdot}1444 + 1{\cdot}5857s + 4{\cdot}8s^2 + 0{\cdot}6429r^2. \tag{14}$$

The maximum static deflection under gravity is

$$\delta = mgL^4/EI(M_S/K), \tag{15}$$

where,

$$M_S = (0{\cdot}75 + 3s)(0{\cdot}3 + 2s). \tag{16}$$

From equations (12) and (15),

$$\omega = C(g/\delta)^{1/2} \tag{17}$$

where

$$C = (M_S/M_D)^{1/2}. \tag{18}$$

Equations (18), (14) and (16) show that C is independent of the rotation parameter α. Hence the value of C calculated for non-rotating beams also holds for rotating beams. However, the value of δ depends upon α^2. At higher rotation speeds, δ decreases and hence ω increases.

From equations (12), (15), and (18)

$$C^2 = (M_S/M_D) = \omega^2\delta/g \tag{19}$$

and

$$C_{BE}^2 = (M_S/M_D)_{BE} = \omega_{BE}^2\delta_{BE}/g. \tag{20}$$

Hence,

$$C = C_{BE}(\omega/\omega_{BE})(\delta/\delta_{BE})^{1/2}. \tag{21}$$

which is the same as equation (2).

3. NUMERICAL RESULTS

Table (1) gives a comparison of the values of C for Timoshenko beams of different slenderness ratios (L/r). The value of E/G is taken as 2·6 and $K = 5/6$. The values calculated from equation (18), using equations (14) and (16) and the values presented by

TABLE 1

Comparison of C values

L/r	C eqn (18)	C ref. [2]
∞	1·2483	1·2431
100	1·2479	1·2424
50	1·2469	1·2405
20	1·2403	1·2325
16·7	1·2371	1·2327
14·3	1·2335	1·2389

Bert are also shown in Table 1. The correlation between the two sets of results is good, the maximum difference being 0·6%.

4. CONCLUSIONS

The non-dimensional constant C in Bert's formula (Equation (1)) relating the fundamental frequency to the maximum static deflection under gravity has been shown to be applicable to both rotating and non-rotating Timoshenko beams. While the value of C is the same for both beams, the value of the fundamental frequency will depend on the angular speed of rotation through the value of δ.

REFERENCES

1. C. W. BERT 1993 *Journal of Sound and Vibration* **162**, 547–557. Relationship between fundamental natural frequency and maximum static deflection for various linear systems.
2. C. W. BERT 1993 *Journal of Sound and Vibration* **186**, 691–693. Effect of finite thickness on the relationship between fundamental natural frequency and maximum static deflection of beams.
3. R. O. STAFFORD and V. GIURGIUTIU 1974 *International Journal of Mechanical Sciences* **17**, 719–727. Semi-analytic methods for rotating Timoshenko beams.